§7.1

2. For each matrix, find a basis for each generalized eigenspace of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of A.

a) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \lambda = 2$ (with multiplicty 2) $\quad K_\lambda = N(A-2I) \quad \dim K_\lambda = 2$

$A - 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \qquad (A-2I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (A-2I)v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (A-2I)^2 v = 0$

$\Rightarrow J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad \beta = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

Rmk: cf. Thm 7.4, 7.2

c) $A = \begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 1$ (with multiplicity 1) $\quad \lambda_2 = 2$ (with multiplicity 2)

$A - 2I = \begin{pmatrix} 9 & -4 & -5 \\ 21 & -10 & -11 \\ 3 & -1 & -2 \end{pmatrix} \qquad (A-2I)^2 = \begin{pmatrix} -18 & 9 & 9 \\ -54 & 27 & 27 \\ 0 & 0 & 0 \end{pmatrix}$

$v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad (A-2I)v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (A-2I)^2 v = 0$

Similarly for $\lambda_1 = 1$, we have $v = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ (which is easier)

$\Rightarrow J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \beta = \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$

3. For the linear operator T, find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T.

a) T is the linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = 2f(x) - f'(x)$

Solution: $\beta = \{1, x, x^2\}$

$$[T]_\beta = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

Proceed as above,

$$J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \beta' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ 0 \\ 2 \end{pmatrix} \right\}$$

Rmk: Do the rest problems yourselves

§7.2

2. T a lin. operator on a fin dim v.s V s.t the char. polynomial of T splits. Suppose that $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = -3$ are distinct eigenvalues of T dot diagrams are as follows, Find ~~the~~ the Jordan canonical form of T.

$\lambda_1 = 2$ $\qquad\qquad$ $\lambda_2 = 4$ $\qquad\qquad$ $\lambda_3 = -3$

Solution:

$$J_1 = \left(\begin{array}{ccc|cc|c} 2 & 1 & & & & \\ & 2 & 1 & & & \\ & & 2 & & & \\ \hline & & & 2 & 1 & \\ & & & & 2 & \\ \hline & & & & & 2 \end{array}\right) \qquad J_2 = \left(\begin{array}{ccc|c} 4 & 1 & & \\ & 4 & 1 & \\ & & 4 & \\ \hline & & & 4 \end{array}\right) \qquad J_3 = \begin{pmatrix} -3 & \\ & -3 \end{pmatrix}$$

$$J = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix}$$

4. Find a Jordan canonical form J and an invertible matrix Q s.t. $J = Q^{-1} A Q$

a) $A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix}$

Solution: The characteristic polynomial of A is $-(\lambda - 1)(\lambda - 2)^2$

$$A - 2I = \begin{pmatrix} -5 & 3 & -2 \\ -7 & 4 & -3 \\ 1 & -1 & 0 \end{pmatrix} \qquad (A - 2I)^2 = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

$$v = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \qquad (A - 2I)v = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda = 1$ $\quad \lambda = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad Q = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ 1 & 1 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

b) $A = \begin{pmatrix} 0 & 1 & -1 \\ -4 & 4 & -2 \\ -2 & 1 & +1 \end{pmatrix}$

Solution: Char. polynomial: $-(\lambda - 1)(\lambda - 2)^2$

$$A - 2I = \begin{pmatrix} -2 & 1 & -1 \\ -4 & 2 & -2 \\ -2 & 1 & -1 \end{pmatrix}$$

Notice that this time $\operatorname{rank}(A - 2I) = 1$

So for $\lambda = 2$, $v_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

For $\lambda = 1$. $\quad v = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad Q = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$